COMPUTER SYNTHESIS OF ANAMORPHIC PROJECTION SYSTEMS

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DEFINITIONS, EXAMPLES

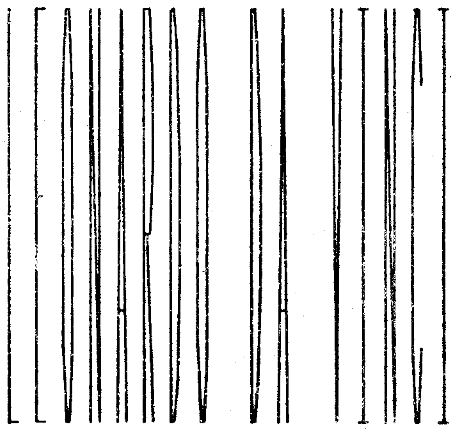
Anamorphic images are those which, when viewed objectively, seem to contain extreme distortions. These distortions become meaningful, however, when the image is viewed subjectively, as from a predetermined vantage point, or when it is reflected in a special mirror. An effective analogy can be drawn to the familiar fun-house mirrors which produce severely distorted reflections. The reflection of normal human torsos in these devices usually appears alternately fat, thin, and curvilinear.

An anamorphic human figure, which might objectively appear distorted to an observer, would then have a reflection that would appear quite normal. Another, somewhat less sophisticated example of an anamorphic image is the word, AMBULANCE, which often appears on the front of such emergency vehicles. The objective message makes little sense, but when viewed by an automobile driver through a rear-view mirror, the subjective message is quite clear.

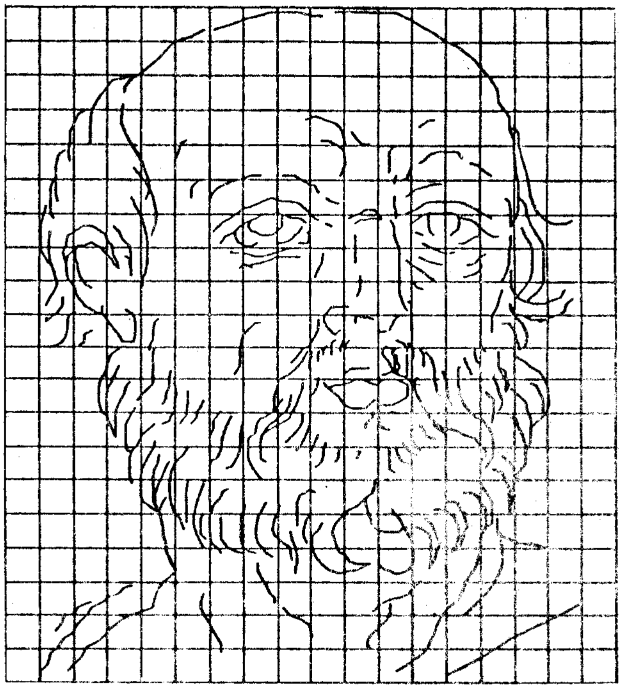
HISTORICAL BACKGROUND

One of the earliest documented studies of anamorphic images and anamorphic projection systems was conducted by Leonardo da Vinci. His sketches reveal that he investigated the properties of images projected on oblique surfaces. Such images, when viewed from a point perpendicular to their plane, are often unrecognizable. However, when the angle between the observer and image plane becomes very small (5 degrees or less), the substance of the form is coherent. (Figure 1.)

NOTE: Tilt the anamorphic illustrations for a less distorted view--Below: The name of Leonardo.



ABOVE: Figure 1 -- (View from 5° horizontal)

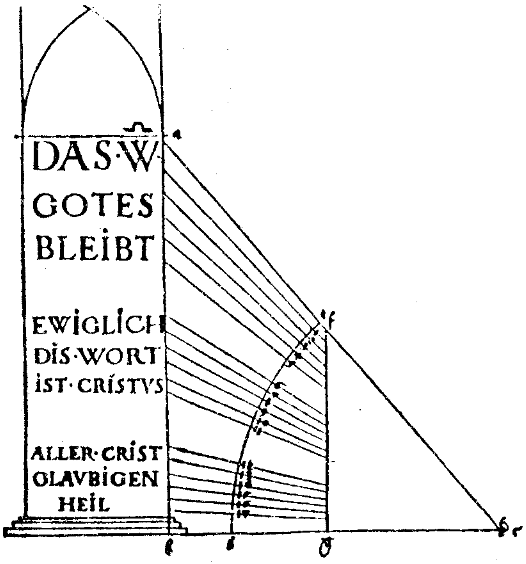


ABOVE: Undistorted portrait of a man on a grid. This source drawing is later revealed within varied projections throughout this paper.

The study of perspective drawing techniques received a great deal of attention during the Italian Renaissance. Alberti, among others, sought to quantify and describe the principles and techniques surrounding perspective drawing. His treatise, "De Pictura", developed many theories of perspective foreshortening based on the angle of vision subtended in the eye by the object being viewed.

Later, Durer expanded this concept to the conclusion that: equal angles of vision = apparently equal size of objects being viewed. An example of this is the apparent size of a dime held at arm's length and the full moon -- they both appear to be the same size. It is only the presence of familiar objects in a visual scene that permits accurate interpretation. Accordingly, Durer presented a scheme for placing inscriptions on high walls: the size of the letters increase as they are placed upward on the wall. When viewed from a point near the base of the wall, all the letters subtend the same angle of vision and therefore appear to be the same size. (Figure 2.)

(See Figure 2 on the next page.)

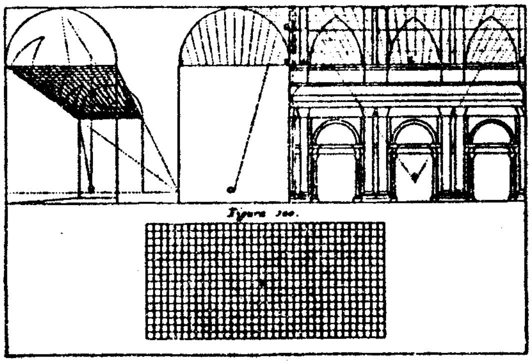


ABOVE: Figure 2 -- Wall Lettering (Durer)

The understanding of angles of vision and the concurrent concept of visual rays (imaginary rays which carry the image to the eye), was the most important concept in the development of anamorphic images. The realization was made that, if these visual rays could carry an image forward to the eye, then one might also extend them backward, away from the observer, to an intersection with any form of surface. Then, using the visual rays as a guide, the image can be transferred to this back surface. Objectively, the projected image will most probably appear quite distorted. However, when viewed from the original point used for the construction, it will regain its natural form. The explanation is simply that, from this point, the two images (original and projected), are identical in terms of visual rays and angles of vision subtended.*

CONSTRUCTING ANAMORPHIC IMAGES

The concept of constructing such images is simple enough, but the practical implementation becomes more complex, especially since the main component, visual rays, are imaginary. The problems surrounding the implementation of this procedure were overcome by Andrea Pozzo and described fully in his treatise, "Perspectivae Pictorum atque Architectorum", which appeared in 1693. The technique described by Pozzo, which related to painting on architectural domes and vaults, is called "Quadratura". (Figure 3)



ABOVE: Figure 3 -- Quadrature (by Andrea Pozzo)

A rectangular grid of ropes or strings is placed across the base of the vault or dome. This network corresponds to another grid which has been placed on a sketch of the desired painting. Then, strings are stretched from a central viewing point through the intersection of the grid and continued to the surface of the vault. In this manner, the grid was transferred to the curved surface, and, the sketch was copied over square by square to the distorted grid. Similar techniques were also used in the production of frescoes on oblique surfaces. The end results were images which seemed to float free of their surfaces and so served to visually extend the limits of the architectural space.

During the seventeenth century a number of French mathematicians expanded on this knowledge of projection systems and added to it the newly discovered principles of optics. The net result was the development of complex systems for anamorphoses utilizing, among other things, the reflecting cone and cylinder. The principal work on this subject, entitled "La Perspective Curieuse", was published in 1638 by Jean-Francois Niceron. This document presented the geometrical basis for the construction of these anamorphoses and was quickly assimilated by both artists and mathematicians. As a result, the next hundred years enjoyed a vogue of anamorphic engravings and paintings, during which a number of interesting, if not artistic, works evolved.

ANAMORPHIC IMAGES VIA A DIGITAL COMPUTER

This paper presents some of the techniques relating to the construction of anamorphic images and the implementation of those techniques on a digital computer. The various forms of anamorphic images mentioned above have been classified by this author into three categories to simplify their explanations in this paper. These categories are:

1. PROJECTION, for those images which require only observation from a predetermined point to appear normal;

2. REFLECTION, for those images which require specially shaped mirrors in which their reflection appears normal; and

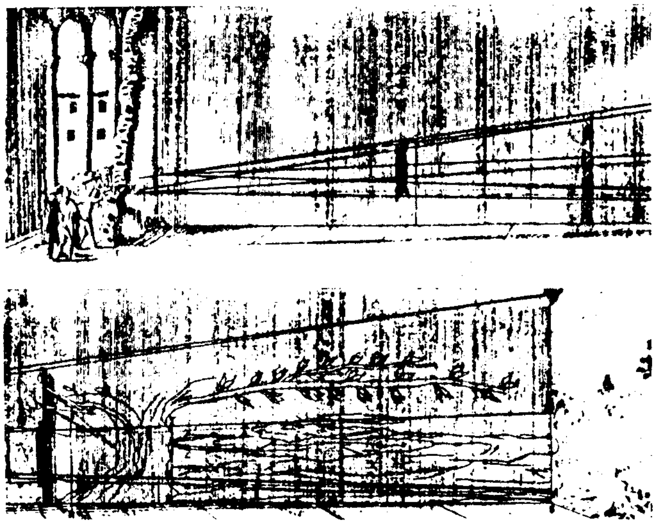
3. CONSTRUCTION, for those images which must be transferred to the surface of a constructed solid to appear natural.

In many respects, the CONSTRUCTION category is a subset of the PROJECTION category. In the following pages the various types of anamorphic projection systems will be presented and analyzed in approximate chronological order based on their initial development and use.

PROJECTION ANAMORPHOSES -- PLANAR SURFACES

One of the most interesting projection anamorphoses was painted on a wall in the Monastery of SS. Trinita dei Monti, in Rome, by Maignan in 1642. Figure 4 is an original engraving by the artist illustrating the technique that he used in the construction.

**Subtend - Sub + tendere, from the Latin, to stretch; to extend under, to be opposite to, as each side of a triangle subtends opposite angles.*



ABOVE: Figure 4 -- St. Francis of Paola (Maignan)

The painting, St. Francis of Paola, is located on a corridor wall, and appears quite normal when viewed from the end of the corridor. However, as one proceeds down the corridor, the image becomes increasingly distorted until, when viewed from a point perpendicular to the wall, it is unrecognizable.

Figure 5 (below) illustrates the geometry of this construction. Plane I-I contains the image which is to be painted on surface A-A. In order for the painted image to appear normal to an observer at point "O", it must be constructed according to the linear distortions exhibited on plane B-B. The concept is this: since objects which are more distant from an observer naturally appear smaller (through perspective foreshortening), they must be made larger if they are to appear the same size as closer objects. This construction is very elementary for the computer.

For simplicity, we consider the observer to be located at the origin of a three-dimensional set of cartesian coordinates. Figure 5 is displaying this construction, which is being viewed looking down from the positive y-axis. Then, any visual ray connecting the observer and a point of the source image plane has the equation: Y = M1X where M1, the slope, equals the distance separating the observer and the image divided by any image x-coordinate. The surface on which the image is to be projected makes a known (specified) angle with the z-axis at a distance (DP) which is also specified.

The equation for this line is in the form: Z = M2X + B, where M2 is the known angle and B is the DP distance. Then, through simple analytic geometry:

visual ray: Z = M1X
projection surface: Z = M2X + DP

the intersection:

M1X = M2X + DP

and, the common x-coordinate:

Xc = 1.0 / ((M1 - M2) / - DP)

The y-coordinate, of course, rarely lies in the plane used for this construction. Its value is determined through a system of similar triangles.

The projection of the point on the z-axis:

Z = Tan (θ) * Xc, or,
Z = (DI / any x-coordinate) * Xc

The ratio of similar triangles is:

(any image y-coordinate / DI) =
(the projected y-coordinate / Z)

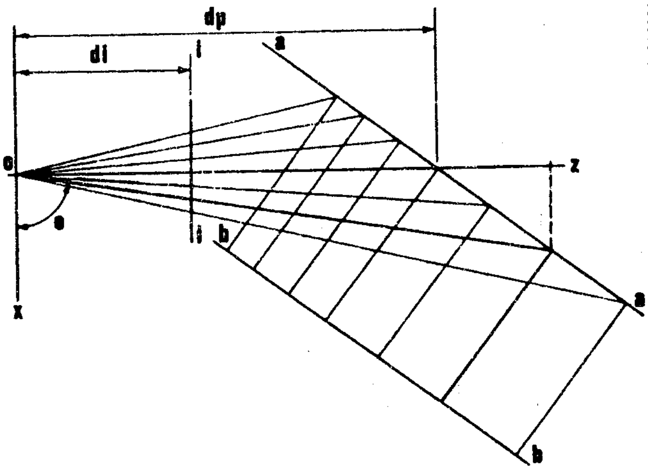
or, simply:

Yc = (any image y-coordinate * Z) / DI

Finally, in order to correctly plot the anamorphic image, it must appear as on plane B-B. This is done by substituting the distance from the z-axis along the projection surface to the visual ray intersection for the x-coordinate, with careful attention to the appropriate sign of the coordinate. This procedure allows for the option of a second projection surface at any specified angle, and oriented in the negative x-direction.

Figures 6 and 7 (see next page) are examples of plotted output from this program illustrating projections on one and two surfaces respectively.

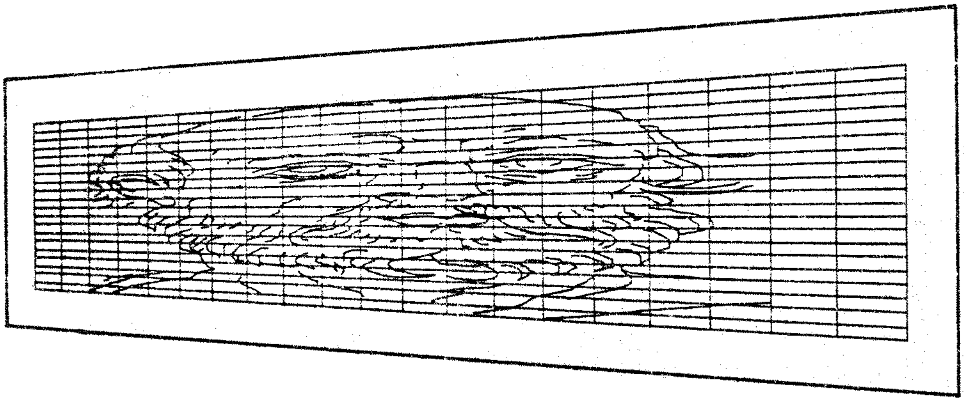
BELOW: Figure 5 -- Geometry of Planar Projection



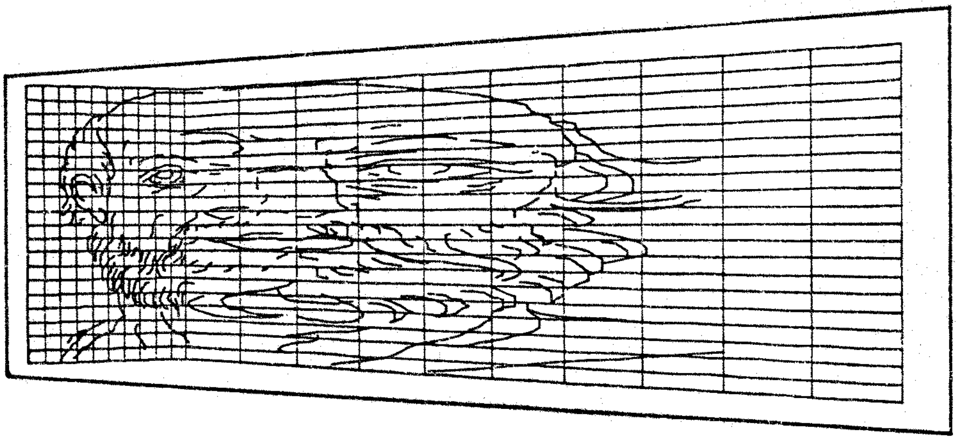
PROJECTION ANAMORPHOSES -- CURVILINEAR SURFACES

There are many examples of this type of anamorphic projection. Paintings and frescoes on vaulted surfaces have been common architectural elements since the forms first emerged. The most successful of these images as illusions of extended architectural space have, as their foundation, the techniques developed by Andrea Pozzo. These techniques, or "quadratura", as previously discussed, were elaborate manual constructions for projecting a rectangular grid onto curvilinear surfaces utilizing a system for simulating visual rays. The projected, distorted grid then became the basis for copying the desired image which had been previously subdivided by an undistorted grid to facilitate the transfer.

(See page 13 for figures 6 and 7.)



AT LEFT: Figure 6 -- Computer Planar Projection: One Surface



AT RIGHT: Figure 7 -- Computer Planar Projection: Two Surfaces

IMPLEMENTING COMPUTATIONAL PROJECTION TECHNIQUES

The computational implementation of this technique is no more complex than the intersection of a line and a circle. The line is the equivalent of any visual ray, and, the circle is equivalent to the vaulted form. Since this process usually yields two points of intersection, it is possible to calculate anamorphoses for both concave and convex surfaces. In the actual implementation of this procedure, it was also possible to include the option of two curvilinear projection surfaces, each either concave or convex, and tangent to each other at the central visual ray.

The intersection of a line and a circle can be solved in many ways. The procedure utilized in this study is based on analytic geometry, and, specifically the Law of Sines. Figure 8 (at right) illustrates schematically the most important aspects of this technique.

The observer is considered to be located at the origin of a cartesian coordinate system. The vaulted surface appears with its center located on the z-axis and is DC units distant from the observer. As before, the diagram in Figure 8 is displayed as viewed from the y-axis. The angle at the observer between the central visual ray (CVR) and any visual ray can be determined by the slope of any line having the end-points of (0,0) and any set of image plane coordinates. The relationship defined by the Law of Sines, which is the radius of the vaulted form divided by the sine of this angle, becomes the important factor for projecting the image point.

Factor = Radius / Sin (ang)

Since: A / Sin (a) = B / Sin (b) (The Law of Sines)

Then, this Factor is also equal to the following:

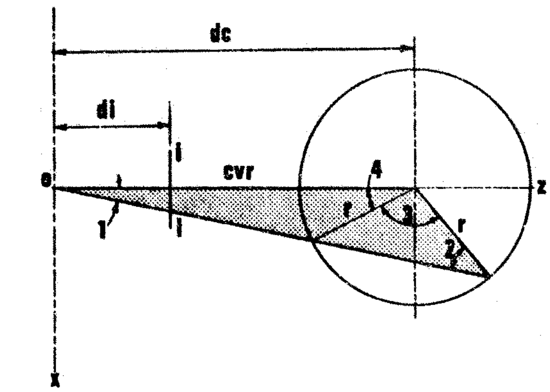
Factor = Dist. to circle center / Sin(ang2)

or,

Sin (ang2) = dist. to circle center / Factor

Ang2 is the interior angle where the visual ray intersects the circumference of the circle. Ang3, which is at the center of the circle, is the

BELOW: Figure 8 -- Geometry of Curvilinear Projection



supplement of Ang and Ang2. The length of the visual ray from the observer to its intersection with the circle, is determined by a similar relationship:

Factor = Length / Sin (ang3), or,
Length = Factor * Sin (ang3)

If, for some reason, the desired projection surface is convex (the opposite curvature of most vaulted forms), the length of the visual ray must be reduced by the hypotenuse of the interior triangle in Figure 8. This is an easy task since all the angles and two sides (radii) are known.

PLOTTING PROJECTED POINTS

The process of plotting this projected point on a flat surface is to theoretically unroll the circular surface by substituting the circumferential distance along the arc for the x-coordinate. This is done by first determining the angle CA in Figure 8, which is either the supplement of Ang3, or, the difference between Ang3 and Ang4, depending upon the curvature of the surface (concave or convex). The ratio of this angle to 360 degrees is also the ratio of the desired circumferential distance to the total circumference. Then, the y-coordinate of the point may be determined by a process of similar triangles:

Y(image) / DI = Y(plot) / dist. to intersection

or,

y(plot) = (Y(image)*dist. to intersection) / DI

This is a process for determining the end-points of a line to be projected onto a circular surface. However, if one is to simply project and plot end-points, the resulting segment will be a chord (relative to Figure 8) and not a true circular projection.

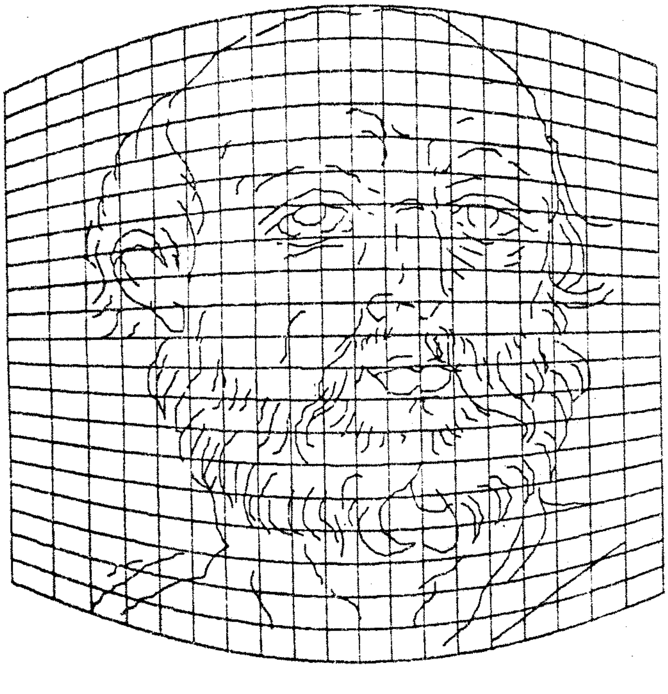


Figure 9 --
Computer Curvilinear Projection: Concave

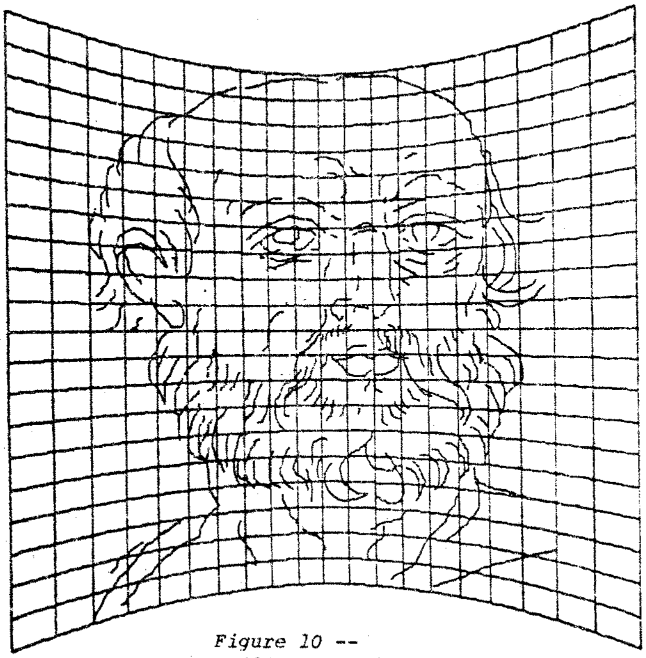


Figure 10 --
Computer Curvilinear Projection: Convex

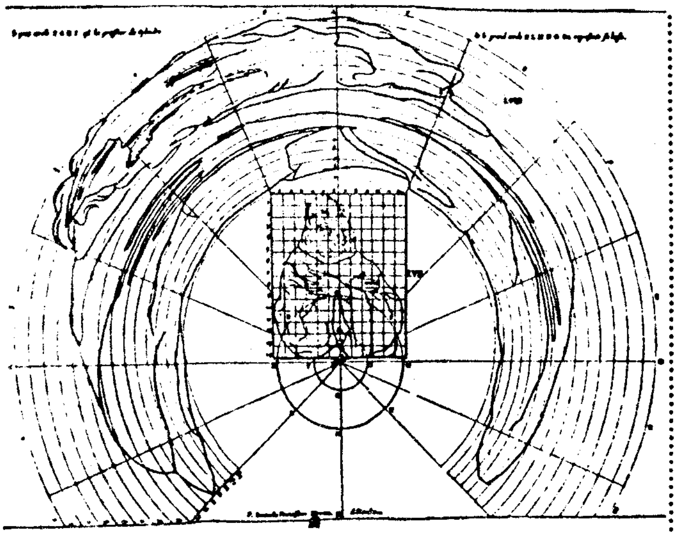
Therefore, in order to approximate a true circular projection, the line to be projected should first be subdivided into many small segments which, when projected individually, will closely approximate the desired results. This subdivision is best accomplished by calculating the angle between the visual rays connecting the observer and the end-points of the line to be projected. This angle is then divided into one-sixth of a degree intervals which are used to calculate the intermediate coordinates in the divided line.

This method is superior to the arbitrary subdivision of each line into a fixed number of segments. Using it, short lines will have few segments and long lines will have many, and, no lines will be subdivided beyond the point of visual relevancy. Three-dimensional angles do not have to be computed for this process because the vaulted surface is only curvilinear in two dimensions.

Figures 9 (bottom, left) and 10 (top, right) illustrate some typical plotted output from this program displaying sample anamorphoses for concave and convex surfaces respectively (the illustrations may be bent toward or away from the observer in an appropriate cylinder to remove the distortion).

REFLECTION ANAMORPHOSES

"La Perspective Curieuse", published by Niceron in 1638, is considered to be the most important documentation of the techniques of reflection anamorphoses. The work was republished in a more extensive form with the title, "Thaumaturgus Opticus", after Niceron's death in 1646. These documents, as well as some earlier and many later ones, notably "Perspective Cylindrique et Conique", by Vaulezard, in 1630, and "La Perspective Pratique", by DuBreuil, in 1649, are primarily responsible for the great vogue of cylinder and cone anamorphoses which followed during the next hundred years.



ABOVE: Figure 11 -- Reflecting Cylinder Anamorphosis (Niceron)

Although some oils were executed by Niceron, the principal anamorphic works for the first half of this period were engravings. These were produced in great quantities and rapidly spread through Europe and England. These engravings supplied the inspiration and the technique for the many anamorphic paintings which began to appear during the early and mid-eighteenth century, notably in the Netherlands and in England, where the works of Henry Kettle are among the best remaining examples of this technique.

REFLECTION ANAMORPHOSES -- THE CYLINDER

Figure 11 (above) is an illustration used by Niceron in his "La Perspective Curieuse". The grid which is superimposed on the portrait assists in communicating the techniques of creating the anamorphosis. The vertical lines seem to become radial, and the horizontal lines seem to become somewhat concentric. Close inspection of the anamorphoses of the horizontal will reveal, however, that they are not circular, but somewhat elliptical. An official geometric description of this shape could be the Limacon of Pascal. The equation for this form is:

$$R = B - (A * \cos(\theta))$$

where:

B is greater than or equal to 2 * A

In this application it is probably more accurate to express this shape as the locus of all points equidistant from a single point via a reflection off a circular form (Figure 12 at right). The single point corresponds to the observer, the circular form is the reflecting cylinder and the locus points occur equidistantly along visual rays defined by the position of the observer and the source image. Another look at Figure 11 (above) should also reveal that the spacing between the projected horizontal lines increases with the distance it is from the center of the cylinder. This relationship is illustrated in more detail in Figure 13 (see next page), where it may be observed that the top of the source image defines visual rays which must travel a longer distance to the anamorphic surface and thus exhibit a greater divergence.

The process of calculating a reflecting-cylinder anamorphosis by computer is somewhat more complicated than the ones previously described for projection, but, readily decomposes into a number of simple and distinct operations.

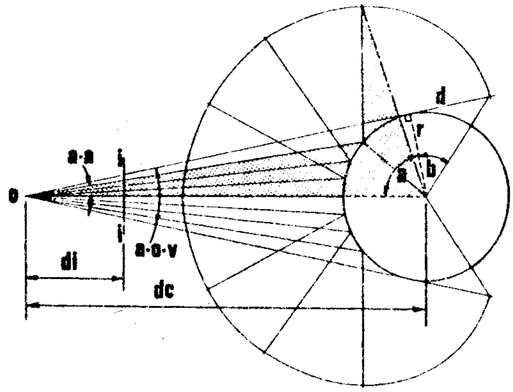
The most obvious approach to pursue after inspecting Niceron's engraving is to somehow express the anamorphic image in terms of polar coodinates relative to the center of the reflecting cylinder. This would require an unusual relationship between the rectangular coordinates of the source image and the desired anamorphic polar coordinates. Instead of normal transformation equations, this approach requires that the x-coordinate of the source image be related in terms of an angle, theta, and that the y-coordinate be expressed in terms of a distance or radius to the cylinder center. The procedure to accomplish this has a number of steps.

First, it is necessary to solve for the total angle of vision of the observer (noted in Figure 12). The right triangle containing the angle "A" is well-defined and all angles and sides can be rapidly calculated. The total distance from the observer to any point on the anamorphic image has to be the same for the construction to work. This distance is an important factor since it controls the reflected angle of vision and the portion of the cylinder's surface which will be utilized by the reflection. An experimental value equal to twice the distance separating the observer and the surface of the cylinder seems to achieve optimal results. Theoretically, any value between one and infinity times this distance would also work. Lower values will decrease the angle of vision and the area of the cylinder surface used to produce the reflection. A value of one is the limit; if used, the cylinder becomes mathematically planar, and the reflection becomes a line.

Once this quantity is chosen, the distance from the observer to the point of tangency between the visual ray and the cylinder may be subtracted to yield the value of "D" in Figure 12. Then, the angle "B" can be calculated, and, twice the sum of angles "A" and "B" yields the total angle of vision relative to the reflection. The next operation is to normalize the range of the x-coordinate on the source image to this angular range. This operation is performed so that an angle of ninety degrees corresponds to an x-value of zero. The extremes of this range then become:

-Xmax = 90 - (Total / 2.0), and,
+Xmax = 90 + (Total / 2.0)

BELOW: Figure 12 -- Reflecting Cylinder Geometry (Plan)



Thus, every x-coordinate on the source image will have a corresponding angle, theta, which will be used in plotting the anamorphic image.

The second half of this process is the determination of the required radius expression for the polar coordinates. This quantity corresponds to the y-coordinate values on the source image. At first glance, the considerations of three-dimensional angles and distances might seem a bit overpowering. Fortunately, reflections, by their very nature, yield a family of similar triangles. Additionally, since all three-dimensional distances for a visual ray bear the same ratio to their plan projections, it is possible to confine the construction to only these two-dimensional projections. The process then becomes much simplified and results in equally correct anamorphic images.

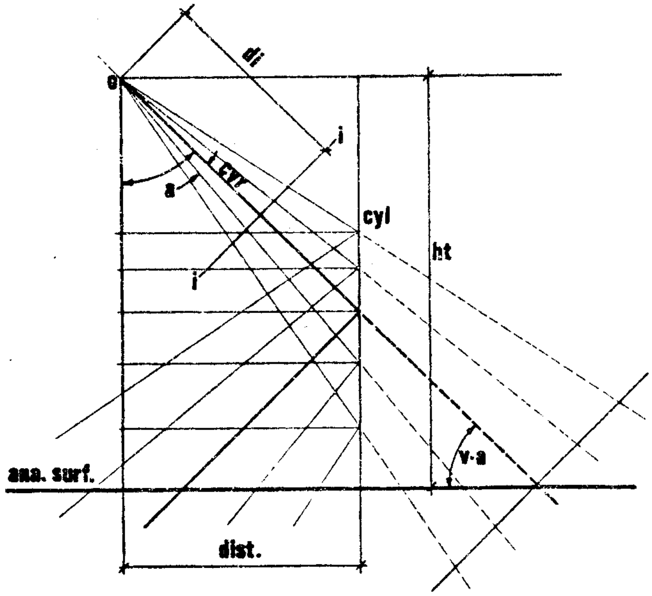
Figure 13 (right) also illustrates that the plan distances from the edge of the reflecting cylinder to the intersection of any visual ray with the anamorphic surface is the same distance which would occur with a transparent cylinder. This concept becomes very helpful in the construction. The plan projection of any visual ray may be determined in the following manner:

First, the vertical angle between the central visual ray and other visual ray is calculated and then appropriately added or subtracted from the angle between the central visual ray and the anamorphic surface.

Next, the tangent of this angle multiplied by the height of the observer will yield the total plan projection of this ray.

The following steps are concerned only with the plan prolections of the visual rays as they appear in Figure 12. (See preceding page.) The intersection of any visual ray with the surface is the next priority operation. This is accomplished by constructing a triangle with vertices at the observer, the intersection of the visual ray and the cylinder, and, the center of the cylinder. The apex angle of this triangle at the observer is the same as the horizontal angle defined by the image plane coordinates. Knowing this angle, the distance to the cylinder center and the radius of the cylinder, it is an easy matter to solve for the remaining sides and angles by applying the Law of Sines. Some care must be taken to account for the two intersections which occur with a line and a circle and to choose the appropriate one. The difference between total plan lengths of the ray and the side of this triangle from the observer to the intersection will be the plan projection of the reflected visual ray. Knowing this length, the radius of the cylinder and the angle between the two (which, by the definition of reflection, is equal to the angle of the previously-solved triangle at the common radius-leg), the Law of Cosines may be applied to yield the required radius-distance to complete the necessary information for the anamorphoses.

As before, the line segments on the source image are subdivided to approximate the true curvilinear character of the cylinder anamorphoses. This subdivision is only necessary in the x-direction (horizontally) because the cylinder is only curvilinear in two dimensions. Figure 14 is an example of typical plotted output calculated by



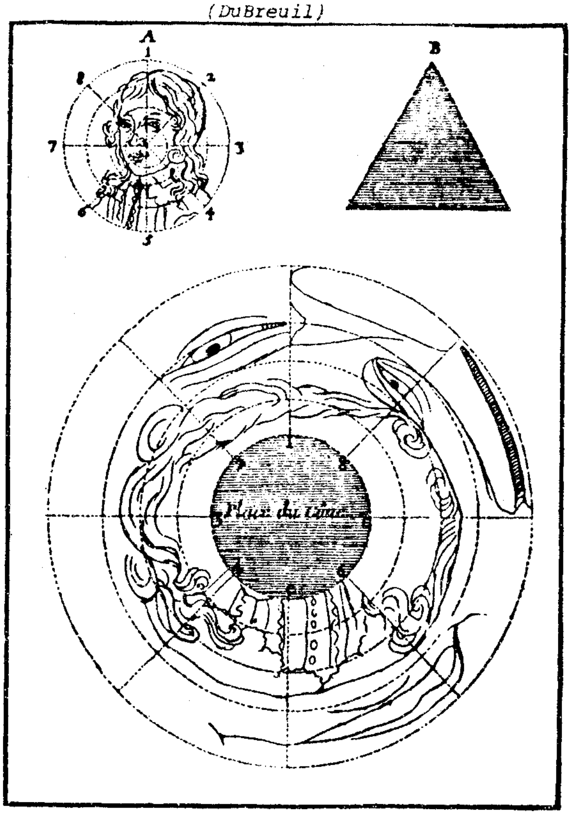
ABOVE: Figure 13 -- Reflecting Cylinder Geometry (Elevation)

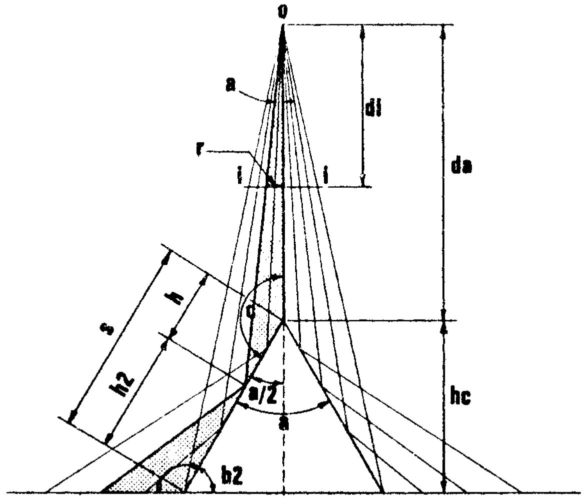
this program. (See the next page for Figure 14.)

REFLECTION ANAMORPHOSES -- THE CONE

Figure 15 is an engraving from "La Perspective Pratique...", published in 1649 by DuBreuil, who was one of Niceron's followers. As one may observe, the process involves inverting and expanding the original image.

BELOW: Figure 15 -- Reflecting Cone Anamorphosis (DuBreuil)





ABOVE: Figure 16 - Reflecting Cone Geometry

(For discussion of Figure 16, see text at right.)

Figure 16 (at left) is a geometric diagram of this process, and, illustrates that the solution is of low difficulty. If the angles of the cone are known as well as the various distances separating the observer, source image and cone, then the calculation of the intersection of the visual ray with the anamorphic surface is relatively simple. There are, however, some important considerations which must not be overlooked in implementing this procedure in a computer program. These are:

1. Visual rays which project directly on the apex of the cone do not reflect. This condition must be checked, and if it occurs, the ray must be moved a small fraction off the apex.

2. Because the reflection surface is curvilinear, the source image lines must be subdivided as before. However, since the anamorphic image is inverted and expanded, lines which occur close to the center of the source image must contain more subdivisions than lines which occur at the extremities. This problem can be overcome by utilizing a sliding scale of subdivision tolerance which ranges from

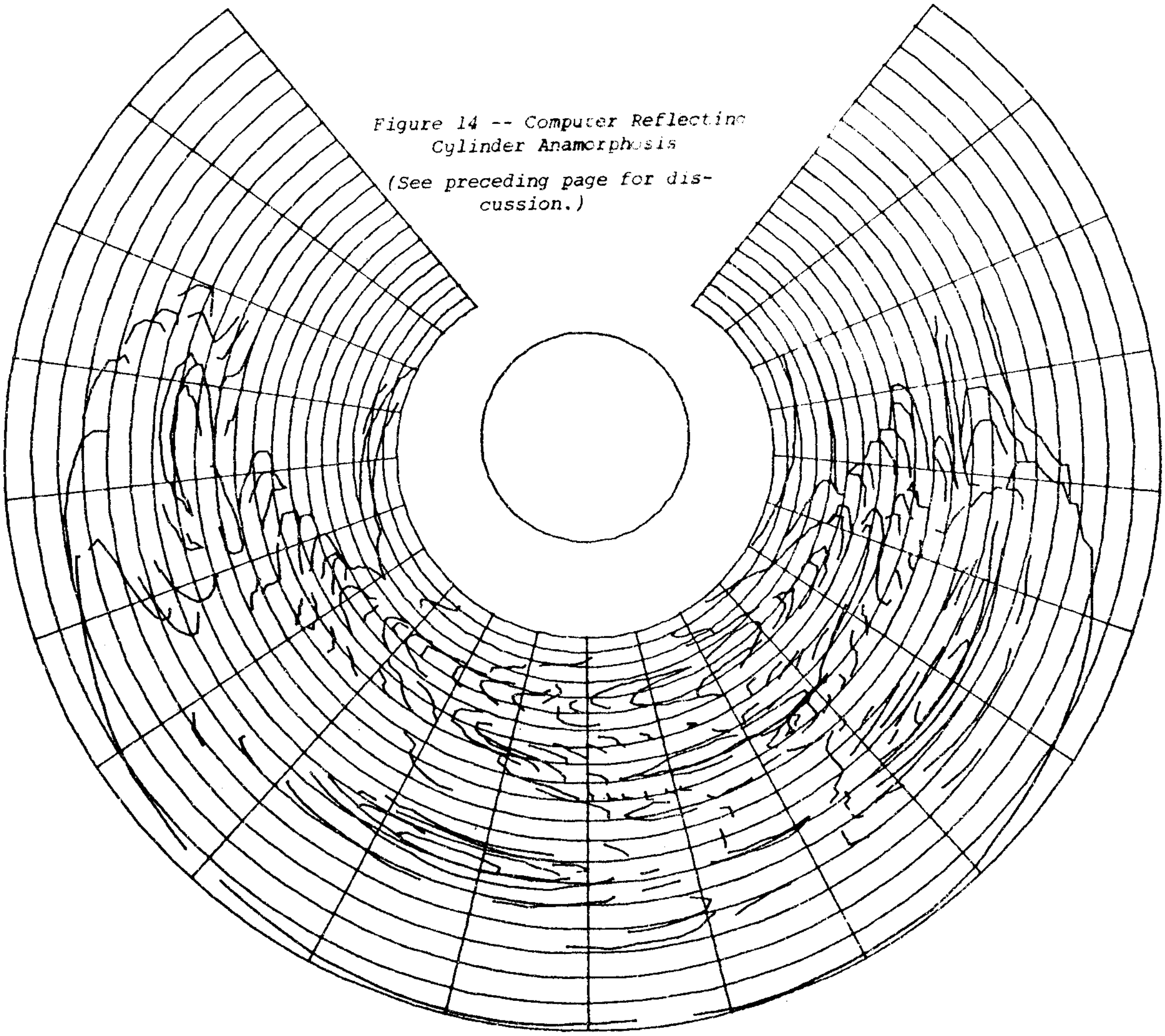


Figure 14 -- Computer Reflecting Cylinder Anamorphosis

(See preceding page for discussion.)

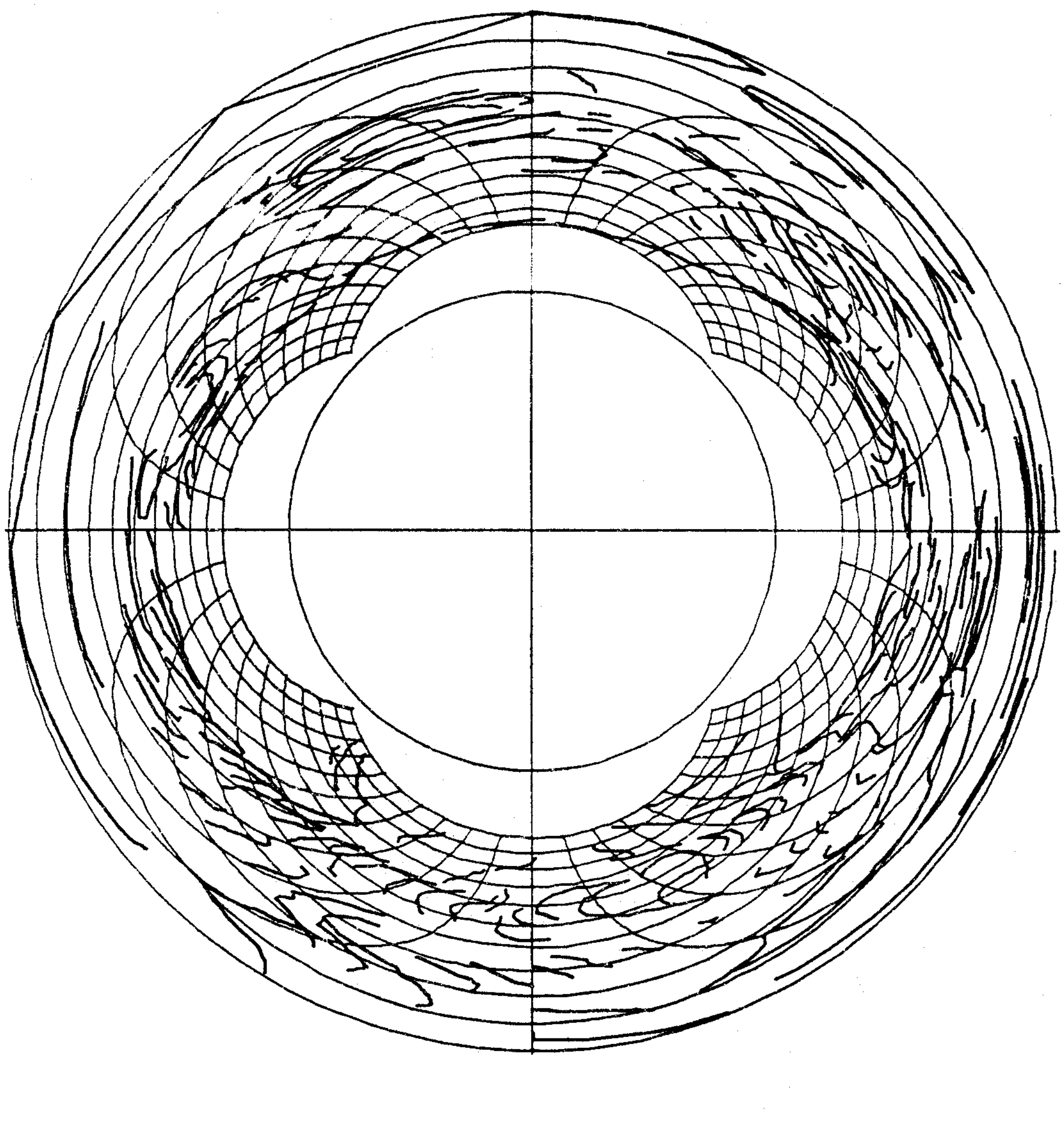
one-sixth of a degree at the extremities to one-sixtieth of a degree at the center. The most logical way to implement this seems to be a binary subdivision process where each line segment on the source image is progressively halved until the remaining segment subtends the appropriate angle for the portion of the image in which it occurs.

The technique utilized in programming the cone anamorphoses proved to be quite efficient. First, the coordinates of the line segments on the source image are converted from rectangular to polar. Then, the radius of each point is used to calculate the angle occurring at the observer between the central visual ray and the visual ray passing through this point (see Figure 16 on the preceding page). Once this angle is known simple techniques of complement, supplement and reflection are used to determine all the angles required by the construction. Some of the lengths of these triangles are known (the distance from the observer to the apex of the cone and the height of the cone). Once again, the Law of Sines is used to calculate the remaining unknown lengths, especially the length of the base projection of the visual ray on the anamorphic surface. The length of this projection is then used as the radius in a new set of polar coordinates which describes the anamorphosis. The angle in this coordinate set is the same one which was calculated for the original point on the source image. Finally these new polar coordinates are reconverted to rectangular for plotting. Figure 17 (below) represents a typical output plot from this program.

CONSTRUCTION ANAMORPHOSES -- THE CONE

The conic-construction anamorphosis is one of the easiest to calculate on a computer. Working on user specifications of cone height and apex angle, it is relatively simple to calculate and plot a planar shape which, when cut out and rolled, will form the desired cone. The radius of this planar shape will be the length of the cone side from apex to base, and the central angle of the wedge-like form will be that angle which subtends an arc of the same length as the base circumference of the cone.

BELOW: Figure 17 -- Computer Reflecting Cone Anamorphosis



As in the cone-reflection anamorphosis just described, the source image coordinates are converted from rectangular to polar, and, the same techniques are utilized to determine the intersection of each visual ray with the side of the cone. The distance from the apex of the cone to this intersection then becomes the radius of the polar coordinate anamorphosis. The angular part of this coordinate is, once again, the original angle of the source coordinates, with the range having been normalized from the original three-hundred and sixty degrees to the value of the central angle which was calculated for the planar cut-out. Figure 18 is an example of the plotted output from this program. If the shape is cut out, rolled into a cone, and, viewed from a point directly above the apex, the image will regain its natural form.

WORK IN PROGRESS

The research presented in this paper represents the preliminary results of a project which is continually expanding. Some of the remaining work, in progress or anticipated, is concerned with the following areas:

- Construction anamorphoses for the pyramid and other common geometric shapes;
- Reflection anamorphoses in which the anamorphic image is non-planar;
- Reflection anamorphoses for sphere and other solid forms;
- Investigations into "transmittance" anamorphoses which will require special optical lenses (curvilinear or faceted) to view the anamorphic image.

SOME APPLICATIONS

Apart from their artistic value and curious novelty, some of the techniques of anamorphoses presented in this paper have had some practical applications in the area of theatre sets. The anamorphic image of the desired scene is plotted and, if necessary, manually colored. The image is photographed onto a slide, which can then be projected at an oblique angle (from the stage wings) onto a backdrop curtain. Because of the extreme angle of the projection, the image regains its natural desired form and provides the appropriate set.

ACKNOWLEDGMENTS

The following illustrations which appear in this paper were reproduced from the book, "Hidden Images", by Leeman, Elffers and Schuyt with the kind permission of the publisher, Harry N. Abrams, Inc., New York: Figures 2, 3, 4, 11, and 15. This book also provided the seed for this project, and I am indebted to the authors and the publisher for its existence.

ABOVE: Detail, "Beauty Adorns Virtue" by Leonardo.

BELOW: Figure 18 -- Computer Cone Construction Anamorphosis